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Filon Trapezoidal Schemes for Hankel Transforms of Orders Zero and One

R. BARAKAT*
Aiken Computation Laboratory, Harvard University
Cambridge, MA 02138, U.S.A.

B. H. SANDLER
Gordon McKay Laboratory, Harvard University
Cambridge, MA 02138, U.S.A.

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Abstract—Algorithms for evaluating zero-order and first-order Hankel transforms using Filon quadrature philosophy are developed in the context of a trapezoidal approximation rather than of a Simpson's rule approximation previously discussed. Unlike the Filon/Simpson algorithm previously developed, the Filon/trapezoidal algorithm tends to saturate, in that increasing the number of quadrature points does not materially increase the accuracy. Numerical examples are given and discussed. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Previous papers [1,2] were devoted to the numerical evaluation of Hankel transforms of orders zero and one,

$$H(r) = \int_a^b h(p) J_n(rp)\, p\, dp, \tag{1.1}$$

using Filon quadrature philosophy. In [1], the Filon approach is outlined in some detail for equation (1.1) with $n = 0$. There, the slowly varying part of the integrand, $h(p)$, is approximated by a quadratic function over the basic quadrate panel. For $n = 1$, see [2]. It was necessary to consider $\bar{h}(p) \equiv ph(p)$ as the basic function to be expressed as a quadratic. As with Filon's original approach to Fourier integrals, the errors incurred in equation (1.1) are proportional to the derivatives of $h(p)$ and $\bar{h}(p)$ themselves rather than to the whole integrand, and hence, are relatively independent of r.

In many areas, we do not require great accuracy for the Hankel transforms, but need to maintain a given accuracy more or less uniformly, independent of the magnitude of r. The purpose of the present communication is to develop the trapezoidal version of the Filon-Hankel approach. In [1,2], the scheme is really a generalization of Simpson's rule, since we are employing

*Deceased.

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a quadratic approximation of $h(p)$ and $\bar{h}(p)$ over the panels. The present approach is essentially a generalization of the trapezoidal quadrature scheme, since we are approximating $h(p)$ and $\bar{h}(p)$ by a linear function over the panels.

2. TRAPEZOIDAL ALGORITHM/ZERO-ORDER TRANSFORM

Consider the integral

$$H_k(r) = \int_{p_k}^{p_{k+1}} h(p) J_0(rp)\, p\, dp, \tag{2.1}$$

for two points p_{k+1} and p_k, where

$$p_{k+1} - p_k = \delta.$$

The points p_k, where $k = 0, 1, \ldots, N$, are a subset of $[a, b]$. Approximate $h(p)$ by a straight line between p_k and p_{k+1}

$$h(p) = A + Bp. \tag{2.2}$$

It follows that

$$A = \frac{1}{\delta}(p_{k+1} h_k - p_k h_{k+1}), \tag{2.3}$$

$$B = \frac{1}{\delta}(h_{k+1} - h_k), \tag{2.4}$$

where $h(p_k) \equiv h_k$.

Upon setting $u = h(p)$ and $dv = J_0(rp) p\, dp$, we integrate $H_k(r)$ by parts; the end result is

$$H_k(r) = \frac{1}{r}[h_{k+1} J_1(rp_{k+1}) p_{k+1} - h_k J_1(rp_k) p_k] - \frac{1}{r^3}\left[h'_{k+1} S_0(rp_{k+1}) - h'_k S_0(rp_k)\right], \tag{2.5}$$

where we have used

$$\int_0^x y J_0(y)\, dy = x J_1(x). \tag{2.6}$$

The numerical evaluation of the function

$$S_0(x) \equiv \int_0^x y J_1(y)\, dy \tag{2.7}$$

is discussed at some length in the Appendix of [1], to which we refer. Since

$$h'_{k+1} = h'_k = \frac{1}{\delta}(h_{k+1} - h_k), \tag{2.8}$$

then equation (2.5) reduces to

$$H_k(r) = \frac{1}{r}[h_{k+1} J_1(rp_{k+1}) - h_k J_1(rp_k)] - \frac{1}{\delta r^3}(h_{k+1} - h_k)[S_0(rp_{k+1}) - S_0(rp_k)]. \tag{2.9}$$

To evaluate the integral over $[a, b]$, i.e., equation (1.1), divide $[a, b]$ thusly

$$b = a + N\delta. \tag{2.10}$$

Hence,

$$H(r) = \sum_{k=0}^{N} H_k(r). \tag{2.11}$$

The final result is

$$H(r) = \frac{1}{r}[h_{N+1} J_1(rp_N) - h_0 J_1(rp_0)] - \frac{1}{\delta r^3} \sum_{k=0}^{N} (h_{k+1} - h_k)[S_0(rp_{k+1}) - S_0(rp_k)]. \tag{2.12}$$

This is the basic formula for the Filon/trapezoidal scheme for zero-order Hankel transforms.

3. NUMERICAL EXAMPLE FOR THE ZERO-ORDER CASE

As with the Filon/Simpson algorithm for the zero-order Hankel transform, we consider as in [1]

$$h(p) = \frac{2}{\pi}\left[\arccos(p) - p\left(1-p^2\right)^{1/2}\right], \qquad 0 \le p \le 1, \tag{3.1}$$

$$H(r) = \left[\frac{2J_1(r)}{r}\right]^2, \qquad 0 \le r < \infty, \tag{3.2}$$

as our test case.

Unlike the Filon/Simpson algorithm, the Filon/trapezoidal algorithm tends to saturate, in that increasing N does not materially increase the accuracy of the quadrature. Of course, this is not surprising, because we are now approximating $h(p)$ by straight lines rather than by quadratic. In the context of our numerical example, perhaps the best way to see this is to fix r while varying N, examining the absolute error

$$\left|H(r)_{\text{exact}} - H(r)_{\text{computed}}\right|. \tag{3.3}$$

Some typical values of the absolute error are displayed in Table 1. This table does not require any detailed comment.

Table 1. Absolute error of Filon/trapezoidal calculations for zero-order case, example in equations (3.1) and (3.2).

r	$N = 100$	$N = 200$	$N = 300$	$N = 400$
6	3.727E−06	9.202E−07	4.069E−07	2.282E−07
12	3.063E−06	7.722E−07	3.447E−07	1.945E−07
18	2.099E−06	5.418E−07	2.443E−07	1.386E−07
24	1.083E−06	2.928E−07	1.343E−07	7.690E−08
30	1.901E−07	6.871E−08	3.427E−08	2.048E−08
40	8.976E−07	2.354E−07	1.068E−07	6.082E−08
50	5.229E−07	1.199E−07	5.167E−08	2.861E−08
60	2.053E−07	6.985E−08	3.411E−08	2.015E−08
80	2.544E−07	4.877E−08	1.941E−08	1.026E−08

In evaluating the various J-Bessel functions, we employed Mason's algorithm [3], which is probably the most accurate currently available.

4. TRAPEZOIDAL ALGORITHM/FIRST-ORDER TRANSFORM

Analogous to equation (2.1), we write

$$H_k(r) = \int_{p_k}^{p_{k+1}} \bar{h}(p) J_1(rp)\, dp, \tag{4.1}$$

where

$$\bar{h}(p) \equiv h(p)p. \tag{4.2}$$

Now, set $u = \bar{h}(p)$ and $dv = J_1(rp)\,dp$. Upon integrating by parts, we obtain

$$\begin{aligned} H_k(r) &= -\frac{1}{r}\left[\bar{h}\left(p_{k+1}\right) J_0\left(rp_{k+1}\right) - \bar{h}(p_k) J_0(rp_k)\right] \\ &+ \frac{1}{r^2}\left[\bar{h}'(p_{k+1}) R_0(rp_{k+1}) - \bar{h}'(p_k) R_0(rp_k)\right], \end{aligned} \tag{4.3}$$

where we have used the indefinite integral

$$\int^{x} J_1(y)\,dy = -J_0(x) \tag{4.4}$$

and

$$R_0(x) \equiv \int_0^{x} J_0(y)\,dy. \tag{4.5}$$

See Appendix A of [2] for the numerical evaluation of this function.

As in Section 2, we can sum the various panels, leading to

$$H(r) = -\frac{1}{r}\left[\bar{h}_N J_0\left(rp_N\right) - \bar{h}_0 J_0\left(rp_0\right)\right] + \frac{1}{\delta r^2}\sum_{k=0}^{N}\left(\bar{h}_{k+1} - \bar{h}_k\right)\left[R_0\left(rp_{k+1}\right) - R_0\left(rp_k\right)\right]. \tag{4.6}$$

This is the basic formula for the Filon-trapezoidal scheme for first-order Hankel transforms.

5. NUMERICAL EXAMPLE FOR THE FIRST-ORDER CASE

Let us consider the example [4]

$$\bar{h}(p) = p\left(1 - p^2\right)^{1/2}, \qquad 0 \le p \le 1, \tag{5.1}$$

$$H(r) = \frac{\pi J_1^2(r/2)}{2r}, \qquad 0 \le r < \infty. \tag{5.2}$$

The numerical results are summarized in Table 2. As with the corresponding algorithm in Section 2, this algorithm also tends to saturate as N increases.

Table 2. Absolute error of Filon/trapezoidal calculations for first-order case, example in equations (5.1) and (5.2).

r	$N = 100$	$N = 200$	$N = 300$	$N = 400$
6	2.764E−04	9.789E−05	5.330E−05	3.463E−05
12	2.186E−04	7.801E−05	4.260E−05	2.772E−05
18	1.769E−04	6.422E−05	3.527E−05	2.302E−05
24	1.360E−04	5.088E−05	2.822E−05	1.851E−05
30	9.406E−05	3.715E−05	2.096E−05	1.386E−05
40	1.129E−04	4.214E−05	2.331E−05	1.526E−05
50	1.029E−04	3.578E−05	1.927E−05	1.243E−05
60	7.152E−05	2.157E−06	1.093E−05	6.818E−06
70	2.863E−05	4.011E−05	9.669E−07	2.174E−07
80	1.479E−05	1.215E−05	7.884E−06	5.537E−06

6. SUMMARY

The Filon/trapezoidal scheme is not meant to be a direct competitor to the Filon/Simpson scheme. The main purpose of this quadrature scheme is to maintain a given accuracy (provided it is not too extreme) more or less uniformly, independent of the magnitude of the independent variable. An added advantage of this scheme is its speed of execution, an important aspect for such problems as beam propagation in an inhomogeneous or random medium, where the integral must be computed a large number of times. Reference is made to [5] for invention of the Hankel transforms using the sampling expansion in connection with the Filon/Simpson and the Filon/trapezoidal schemes.

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